ISRAEL JOURNAL OF MATHEMATICS, Vol. 58, No. 3, 1987

UNIVERSAL WCG BANACH SPACES AND UNIVERSAL EBERLEIN COMPACTS

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ABSTRACT
We prove that for cardinals τ satisfying $\tau^{\omega} = \tau$ and for $\tau = \omega_1$, there do not exist universal Eberlein Compacts of weight τ, or universal WCG spaces of density character τ. If τ is a strong limit cardinal of countable cofinality such universal spaces do exist. Thus under GCH universal spaces exist for τ iff $\operatorname{cof}(\tau) = \omega$.

1. Introduction

A compact Hausdorff space is called an Eberlein Compact (EC) if it is homeomorphic to a weakly compact subset of some Banach space. A Banach space is called Weakly Compactly Generated (WCG) if it is spanned by some weakly compact subset.

In [BRW] the authors ask whether, for a given cardinal τ, there exists an EC of topological weight τ, universal for all EC's of weight τ, and whether there exists a WCG space of density character τ, universal for all WCG spaces of density character τ. (Problems 1 and 2 in [BRW].)

In this article we answer these questions for certain cardinals. Under GCH our results give a complete answer to these problems. The results are as follows:

THEOREM A. *If the cardinal τ satisfies either $\tau^{\omega} = \tau$ or $\tau = \omega_1$, there is no universal* EC *of weight τ in the sense that there is no* EC, *K, of weight τ so that every* EC *of weight τ is a quotient of a closed subset of K.*

† The research of the second author was supported by a grant from the United States–Israel Binational Science Foundation and the Fund for the Promotion of Research at the Technion.
Received January 10, 1987

THEOREM B. *If the cardinal τ satisfies either $\tau^{\omega} = \tau$ or $\tau = \omega_1$, there is no universal* WCG *space of density character τ in the sense that there is no* WCG *space X of density character τ, so that every* WCG *space of density character τ is isomorphic to a subspace of a quotient of X. In fact such X cannot even be universal for all reflexive spaces of density character τ.*

THEOREM C. *If τ is a strong limit cardinal of countable cofinality, there is an* EC $\mathcal{U}$, *of weight τ, universal in the sense that every* EC *of weight τ embeds as a retract of $\mathcal{U}$.*

THEOREM D. *If τ is a strong limit cardinal of countable cofinality, there is a* WCG *space B, of density character τ, universal in the sense that every* WCG *space of density character τ is isometric to a norm one complemented subspace of B.*

To prove Theorems A and B we develop in section 3 an ordinal index for EC's generalizing the Szlenk index introduced in [S]. The proofs then depend on the constructions, in sections 4 and 5, of EC's of weight τ whose indices are unbounded in τ^+. The proofs of Theorems A and B (modulo the results of sections 3–5) are given in section 2. Theorems C and D will be proved in section 6.

Under GCH every infinite cardinal satisfies either $\tau^{\omega} = \tau$ (when $\mathrm{cof}(\tau) > \omega$), or $\mathrm{cof}(\tau) = \omega$ and then it is a strong limit. We thus obtain

THEOREM E. *Under* GCH *universal* EC's *or* WCG *spaces exist for τ iff τ has countable cofinality.*

We shall use standard notations and terminology concerning ordinals and cardinals, topology and Banach spaces. In particular we denote by $\omega(\omega_1)$ the first infinite (uncountable) ordinals, and we identify a cardinal with the first ordinal of its cardinality. For a cardinal τ, we denote by τ^+ its successor. The cofinality of an ordinal α is denoted by $\mathrm{cof}(\alpha)$, and it is the smallest cardinality of a net of ordinals increasing to α. The cardinality of a set Γ is denoted by $|\Gamma|$. A cardinal τ is a strong limit if $2^{\alpha} < \tau$ for all cardinals $\alpha < \tau$.

The weight $w(K)$ of a topological space K is the minimal cardinality of a base for its topology. The density character of K is the minimal cardinality of a dense subset of K. For EC's the density character is equal to the weight.

For a set Γ we denote by $c_0(\Gamma)$ the Banach space of all $f: \Gamma \to \mathbf{R}$ so that $\{\gamma : |f(\gamma)| > \varepsilon\}$ is finite for all $\varepsilon > 0$, with the sup norm. A fundamental theorem of Amir–Lindenstrauss ([AL], see also [G]) says that if X is a WCG space, there is a one-one norm one operator $T: X \to c_0(\Gamma)$ where Γ is a set whose cardinality is the same as the density character of X. In particular every EC, K, is homeomorphic to a weakly compact subset of $c_0(\Gamma)$ with $|\Gamma| = w(K)$.

If Γ_1 is a subset of Γ and $x \in c_0(\Gamma)$ we denote by $x|_{\Gamma_1}$ the element of $c_0(\Gamma)$ given by $x|_{\Gamma_1}(\gamma) = x(\gamma)$ for $\gamma \in \Gamma_1$ and $x|_{\Gamma_1}(\gamma) = 0$ for $\gamma \notin \Gamma_1$. If K is a subset of $c_0(\Gamma)$, we put $K|_{\Gamma_1} = \{x|_{\Gamma_1} : x \in K\}$.

For a Banach space X we denote by $B(X^*)$ the unit ball of X^* with the w^*-topology. If X is WCG, $B(X^*)$ is an EC [L].

An Eberlein compact was called *strong* in [BRW], if it could be embedded as a subset K of $c_0(\Gamma)$, consisting entirely of characteristic functions of (finite) subsets of Γ. We shall also call such a subset of $c_0(\Gamma)$ strong. For such sets we shall identify the characteristic function $\chi_A \in K$ with the set $A \subset \Gamma$, and we shall consider K as a family of finite subsets of Γ. A strong subset of $c_0(\Gamma)$ is called *adequate* if $A \subset B$ and $B \in K$ implies that $A \in K$ too. It is easy to see ([BS], Lemma 2) that an adequate strong subset of $c_0(\Gamma)$ is weakly compact iff it does not contain an infinite increasing chain.

2. Proofs of Theorems A and B

In section 3 we define an ordinal index $i(K)$ for Eberlein Compacts satisfying:

(i) The cardinality of $i(K)$ is at most $w(K)$ — the weight of K.

(ii) If $K \subset H$ then $i(K) \leqq i(H)$.

(iii) If K is a quotient of the EC H, and if K is a strong weakly compact subset of $c_0(\Gamma)$, then $i(K) \leqq i(H)$.

In section 4 we construct, for each cardinal τ such that $\tau^\omega = \tau$, a family of strong weakly compact subsets $\{K_\xi : \xi < \tau^+\}$ of weight τ so that $\sup i(K_\xi) = \tau^+$. In section 5 a similar construction is made for $\tau = \omega_1$.

This proves Theorem A. Indeed, if K were universal with $w(K) = \tau$, then by (i) $i(K) < \tau^+$, yet by (ii) and (iii) we would get $i(K) \geqq \sup i(K_\xi) = \tau^+$, a contradiction.

Theorem B follows from A. To see this assume X were universal. Given any WCG space Y of density character τ, we could find a space Z containing Y and an onto linear operator $T: X \to Z$, $\| T \| = I$. But then $B(Y^*)$ is a quotient of $B(Z^*)$, and the latter is homeomorphic, via T^*, to a subset $B(X^*)$. As the weight of $B(X^*)$ is τ, and as every EC, K, of weight τ is homeomorphic to a subset of $B(Y^*)$ for some WCG space Y of density character τ, it follows that $B(X^*)$ is a universal EC contradicting Theorem A.

In fact, by [DFJP] every EC of weight τ embeds into $B(Y^*)$ for some reflexive Banach space Y of density character τ. Thus X cannot even be universal for reflexive Banach spaces.

3. The index

The definition of the index, $i(K)$, of an EC K will be done in two steps. In the first we generalize the definition of the Szlenk index [S] to weakly compact subsets of a Banach space. This definition uses the particular embedding of the set in the Banach space, and is not a topological invariant. The second step is to define $i(K)$ as the infimum of the Szlenk indices over all possible embeddings of K as a weakly compact subset of some Banach space. This clearly makes $i(K)$ a topological invariant — but is very difficult to estimate. The main part of this section is the definition of another index, $j(K)$, for weakly compact subsets of $c_0(\Gamma)$, which is easier to compute, and which gives a lower bound for $i(K)$, i.e., $j(K) \leqq i(K)$. In fact $j(K) = i(K)$ for strong weakly compact sets.

Let K be a weakly compact subset of a Banach space, and let X be the closed linear span of K. For each $\varepsilon > 0$, define the subset $P_1(K, \varepsilon)$ of K as follows: $x \in P_1(K, \varepsilon)$ iff there are sequences $(x_n) \subset K$, $(x_n^*) \subset B(X^*)$ so that $x_n \overset{\omega}{\to} x$, $x_n^* \overset{\omega^*}{\to} 0$ and $\limsup x_n^*(x_n) \geqq \varepsilon$. As X is WCG, $B(X^*)$ with its ω^*-topology is an EC, hence by the Eberlein–Smulian Theorem both K and $B(X^*)$ are sequentially compact. This allows the diagonalization as in [S] and proves that $P_1(K, \varepsilon)$ is a closed subset of K. By a theorem of Namioka [N], the identity map from K with its weak topology to K with the norm topology is continuous in a dense (and G_δ) subset of K. As the identity is clearly discontinuous at points of $P_1(K, \varepsilon)$, the latter is nowhere dense in K.

Define now inductively $P_{\alpha+1}(K, \varepsilon) = P_1(P_\alpha(K, \varepsilon), \varepsilon)$, and $P_\alpha(K, \varepsilon) = \bigcap_{\beta<\alpha} P_\beta(K, \varepsilon)$ when α is a limit ordinal. The sets $P_\alpha(K, \varepsilon)$ form a strictly decreasing family of closed subsets of K, thus there is an ordinal α, whose cardinality is at most $w(K)$, the weight of K, so that $P_\alpha(K, \varepsilon) = \varnothing$. The ε-Szlenk index of K is defined to be $s(K, \varepsilon) = \sup\{\alpha : P_\alpha(K, \varepsilon) \neq \varnothing\}$, and the Szlenk index of K is defined to be $s(K) = \sup\{s(K, \varepsilon) : \varepsilon > 0\}$. Thus $s(K, \varepsilon) \leqq s(K) < w(K)^+$.

DEFINITION. Let K be an EC. The index $i(K)$ of K is defined as $i(K) = \inf\{s(\varphi(K))\}$, where the infimum is taken over all homeomorphic embeddings φ of K as a weakly compact subset of some Banach space.

We now turn to estimating $i(K)$, and we first note that it is enough to consider homeomorphisms φ of K into $c_0(\Gamma)$. Indeed, if K is a weakly compact subset of some Banach space X, we can assume that K generates X, and then, by the Amir–Lindenstrauss Theorem there is a one-one $T: X \to c_0(\Gamma)$, $\| T \| = 1$. Then $\varphi = T|_K$ is a homeomorphism with respect to the weak topologies, and using the fact that $\| T \| = 1$ one easily checks that $s(T(K), \varepsilon) \leqq s(K, \varepsilon)$ for all $\varepsilon > 0$, i.e., $s(T(K)) \leqq s(K)$.

The remark above reduces the problem of computing $i(K)$ to computation involving only subsets of $c_0(\Gamma)$. And we shall use the following convenient description of $P_1(K, \varepsilon)$, for a weakly compact subset K of $c_0(\Gamma)$, in terms of the coordinates:

$$x \in P_1(K, \varepsilon) \textit{ iff there are sequences } (x_n) \subset K \textit{ and } (\gamma_n) \subset \Gamma,$$

$$\textit{so that } x_n \xrightarrow{\omega} x \textit{ and } \limsup |x_n(\gamma_n)| \geqq \varepsilon.$$

Let now K be a weakly compact subset of $c_0(\Gamma)$ and define

$$j(K, \varepsilon) = \min \left\{ \sup_n s(K|_{\Gamma_n}, \varepsilon) : \Gamma = \bigcup \Gamma_n \text{ and } \Gamma_1 \subset \Gamma_2 \subset \cdots \right\}$$

and $j(K) = \sup\{j(K, \varepsilon) : \varepsilon > 0\}$. Clearly $j(K) \leqq s(K)$ — and the point in defining $j(K)$ is that the stronger inequality $j(K) \leqq i(K)$ holds, as we shall see later. The following Lemma gives the basis for our later inductive proof. Its proof uses a standard saturation argument.

Lemma 3.1. *Let K and H be weakly compact subsets of $c_0(\Gamma)$ and $c_0(\Delta)$ respectively, with $|\Delta| = \tau$. Let $f: K \to H$ be an onto continuous map. Then there are families of increasing subsets $\{\Delta_\xi : \xi < \tau\}$ and $\{\Gamma_\xi : \xi < \tau\}$ of Δ and Γ respectively, so that:*

(1) *$|\Delta_\xi| = |\Gamma_\xi| < \tau$ for all $\xi < \tau$, $\Delta = \bigcup \{\Delta_\xi : \xi < \tau\}$ and $\Delta_\xi = \bigcup_{\beta < \xi} \Delta_\beta$ for limit ordinals ξ. (Hence $\Delta = \bigcup (\Delta_{\xi+1} \setminus \Delta_\xi)$.)*

(2) *For all $x \in K$ and $\xi < \tau$, $x|_{\Gamma_\xi} \in K$.*

(3) *For all $x \in K$ and $\xi < \tau$*

$$f(x|_{\Gamma_\xi})(\delta) = \begin{cases} f(x)(\delta) & \text{if } \delta \in \Delta_\xi, \\ 0 & \text{if } \delta \notin \Delta_\xi. \end{cases}$$

(4) *For each $\eta > 0$ and $\delta \in \Delta$ there is a number $\mu(\delta, \eta) > 0$ so that if $x \in K$ satisfies $|f(x)(\delta)| \geqq \eta$, then $\| x|_{\Gamma_{\xi+1} \setminus \Gamma_\xi} \| \geqq \mu$, where ξ is the unique ordinal so that $\delta \in \Delta_{\xi+1} \setminus \Delta_\xi$.*

Proof. Assuming Δ_ξ and Γ_ξ are already defined, we define inductively sets $A_0 \subset A_1 \subset \cdots$ containing Δ_ξ and $B_0 \subset B_1 \subset \cdots$ containing Γ_ξ, and then take

$\Delta_{\xi+1}=\bigcup A_n$ and $\Gamma_{\xi+1}=\bigcup B_n$. The construction of Δ_1, Γ_1 is similar to this induction step, and for limit ordinals ξ we just take $\Delta_\xi=\bigcup\{\Delta_\beta:\beta<\xi\}$, $\Gamma_\xi=\bigcup\{\Gamma_\beta:\beta<\xi\}$.

Define $A_0=\Delta_\xi\cup\{\delta\}$ and $B_0=\Gamma_\xi$, where $\delta\notin\Delta_\xi$ is arbitrary. Having already defined A_n and B_n, put

$$A_{n+1}=A_n\bigcup\{\text{supports of all } f(x|_{B_n}) \text{ where } x|_{B_n}\in K\}.$$

We now define B_{n+1}. Let $\{\mathcal{U}_\alpha:\alpha<|B_n|\}$ be a base for the topology of $K|_{B_n}$. For each α, let $K_\alpha=\{x\in K: x|_{B_n}\in\mathcal{U}_\alpha\}$, and let L_α be a subset of K_α, $|L_\alpha|\leqq|A_{n+1}|$ so that $f(L_\alpha)|_{A_{n+1}}$ is dense in $f(K_\alpha)|_{A_{n+1}}$.

We now define

$$B_{n+1}=B_n\bigcup\{\text{supports of all } l\in\bigcup L_\alpha\}.$$

As the supports of elements of K and H are countable, and the weight of a subset of $c_0(A)$ is at most $|A|$, we see that $|A_n|$, $|B_n|<|\Delta|$, and the process can be continued until $\Delta=\bigcup\Delta_\xi$, so (1) is satisfied.

We shall check (2) and (3) simultaneously. Fix ξ and let $\delta\in\Delta_{\xi+1}$ and $x\in K$. We need to see that $x|_{\Gamma_{\xi+1}}\in K$, and by the induction hypothesis we then need to check that $f(x|_{\Gamma_{\xi+1}})(\delta)=f(x)(\delta)$ only for $\delta\in\Delta_{\xi+1}\setminus\Delta_\xi$. So fix m so that $\delta\in A_m$. For each fixed $n>m$ consider the set of α's for which $x|_{B_n}\in\mathcal{U}_\alpha$, and using the density of $f(L_\alpha)|_{A_{n+1}}$, find, for each such α, an $x_\alpha\in L_\alpha$ so that $|f(x_\alpha)(\delta)-f(x)(\delta)|\leqq 1/n$. The net (x_α) (ordered by inclusion of the $\mathcal{U}_\alpha$'s) has a convergent subnet, and let x_n be its limit. Then $x_n|_{B_n}=x|_{B_n}$, x_n is supported in B_{n+1}, and $|f(x_n)(\delta)-f(x)(\delta)|\leqq 1/n$. As $\Gamma_{\xi+1}=\bigcup B_n$ we see that $x_n\to x|_{\Gamma_{\xi+1}}$, so $x|_{\Gamma_{\xi+1}}\in K$ and (2) is satisfied. Also by continuity of f, $f(x|_{\Gamma_{\xi+1}})(\delta)=\lim f(x_n)(\delta)=f(x)(\delta)$. Finally, if $\delta\notin\Delta_{\xi+1}$, a similar argument shows that $f(x|_{\Gamma_{\xi+1}})(\delta)=0$.

Part (4) follows immediately from (3) and the continuity of f: If it were false, find x_n so that $|f(x_n)(\delta)|\geqq\eta$, yet $\|x_n|_{\Gamma_{\xi+1}\setminus\Gamma_\xi}\|<1/n$, and let x be a limit point of (x_n). Then $x|_{\Gamma_{\xi+1}}=x|_{\Gamma_\xi}$ and $\delta\notin\Delta_\xi$, hence by (3)

$$\eta\leqq|f(x)(\delta)|=|f(x|_{\Gamma_{\xi+1}})(\delta)|=|f(x|_{\Gamma_\xi})(\delta)|=0,$$

a contradiction.

LEMMA 3.2. *Let K and H be weakly compact subsets of $c_0(\Gamma)$ and $c_0(\Delta)$ respectively, and let $f: K\to H$ be an onto continuous map. Then $j(H)\leqq s(K)$.*

We shall prove the Lemma by establishing the following claim using transfinite induction on $|\Delta|$:

CLAIM. *For every $\eta > 0$ there is an increasing chain $\Delta_1 \subset \Delta_2 \subset \cdots$ with $\Delta = \bigcup \Delta_n$, so that:*

If for some fixed n, $|f(x_j)(\delta_j)| \geqq \eta$ for two infinite sequences $(x_j) \subset K$ and $(\delta_j) \subset \Delta_n$, then there is a subsequence (x_{j_k}) and an infinite sequence $(\gamma_k) \subset \Gamma$ so that $|x_{j_k}(\gamma_k)| \geqq 1/n$.

The Lemma follows immediately from the Claim. Indeed we estimate $j(H)$ by using the chain (Δ_n) given by the Claim. For each fixed η and n, we can use the Claim to prove by transfinite induction that for each α

$$f^{-1}(P_\alpha(H\,|_{\Delta_n}, \eta)) \subset P_\alpha(K, 1/n).$$

Thus $s(H\,|_{\Delta_n}, \eta) \leqq s(K, 1/n) \leqq s(K)$. As n is arbitrary, $j(H, \eta) \leqq s(K)$, and as η is arbitrary, $j(H) \leqq s(K)$.

PROOF OF CLAIM. We use Lemma 3.1 to find sets Δ_ξ and Γ_ξ and for each δ, $\mu(\delta, \eta) > 0$ as in the Lemma.

If $|\Delta| = \omega_1$, the sets $\Delta_{\xi+1} \backslash \Delta_\xi$ are all countable, so we can enumerate them $\Delta_{\xi+1} \backslash \Delta_\xi = \{\delta(\xi, n) : n \in \mathbf{N}\}$. Define now

$$\Delta_n = \{\delta : \delta = \delta(\xi, k) \text{ for some } \xi \text{ and some } k \leqq n, \text{ and } \mu(\delta, \eta) \geqq 1/n\}.$$

Clearly $\Delta = \bigcup \Delta_n$ and $\Delta_1 \subset \Delta_2 \subset \cdots$. If $|f(x_j)(\delta_j)| \geqq \eta$ where $(\delta_j) \subset \Delta_n$ for some fixed n, then by passing to a subsequence we can ensure that there is a fixed $k \leqq n$ so that $\delta_j = \delta(\xi_j, k)$, and in particular the ξ_j's are pairwise different. By the definition of $\mu(\delta, \eta)$ there are $\gamma_j \in \Gamma_{\xi_j+1} \backslash \Gamma_{\xi_j}$ (so, in particular, the γ_j's are different!) with $|x_j(\gamma_j)| = \| x_j\,|_{\Gamma_{\xi_j+1} \backslash \Gamma_{\xi_j}} \| \geqq \mu(\delta_j, \eta) \geqq 1/n$.

For the inductive step, let $\pi_\xi : c_0(\Delta) \to c_0(\Delta_{\xi+1} \backslash \Delta_\xi)$ be the restriction map, and consider $f_\xi = \pi_\xi \circ f : K \to c_0(\Delta_{\xi+1} \backslash \Delta_\xi)$. Applying the inductive hypothesis to f_ξ, we can find for each ξ an increasing chain $\Delta_{\xi,1} \subset \Delta_{\xi,2} \subset \cdots$ so that $\Delta_{\xi+1} \backslash \Delta_\xi = \bigcup \Delta_{\xi,n}$ as in the Claim, and define Δ_n to be those $\delta \in \bigcup \{\Delta_{\xi,k} : \xi < \tau, k \leqq n\}$ for which $\mu(\delta, \eta) \geqq 1/n$.

Again (Δ_n) increases and $\Delta = \bigcup \Delta_n$. Given sequences $(x_j) \subset K$ and $(\delta_j) \subset \Delta_n$ for which $|f(x_j)(\delta_j)| \geqq \eta$, we distinguish two cases:

Case I. There is a ξ so that infinitely many of the δ_j's belong to the same $\Delta_{\xi+1} \backslash \Delta_\xi$ (so, in fact, to $\Delta_{\xi,n}$). By the inductive hypothesis there is a further subsequence (x_{j_k}) and γ_k's so that $|x_{j_k}(\gamma_k)| \geqq 1/n$.

Case II. If each $\Delta_{\xi+1} \backslash \Delta_\xi$ contains only finitely many of the δ_j's, we can pass to a subsequence so that $\delta_j \in \Delta_{\xi_j,n}$ and the ξ_j's are pairwise different. Again by the definition of $\mu(\delta_j, \eta)$ there are $\gamma_j \in \Gamma_{\xi_j+1} \backslash \Gamma_{\xi_j}$ (hence different!) so that $|x_j(\gamma_j)| \geqq \mu(\delta_j, \eta) \geqq 1/n$.

LEMMA 3.3. *If K is a strong weakly compact subset of $c_0(\Gamma)$, then $i(K) \leqq j(K)$.*

PROOF. For strong weakly compact sets $s(K) = s(K, 1)$, so also $j(K) = j(K, 1)$. Thus fix an increasing chain (Γ_n) on which $j(K) = j(K, 1)$ is attained, and define $\varphi : K \to c_0(\Gamma)$ by $\varphi(x)(\gamma) = x(\gamma)/n$ when $\gamma \in \Gamma_n \setminus \Gamma_{n-1}$, and we estimate $s(\varphi(K))$. Fix N, then one easily proves by transfinite induction that for each α

$$\varphi^{-1}(P_\alpha(\varphi(K), 1/N)) \big|_{\Gamma_N} \subset P_\alpha(K \big|_{\Gamma_N}, 1).$$

Thus $s(\varphi(K), 1/N) \leqq s(K \big|_{\Gamma_N}, 1) \leqq j(K)$. As N is arbitrary $s(\varphi(K)) \leqq j(K)$, and thus certainly $i(K) \leqq j(K)$.

THEOREM 3.4. *For every weakly compact subset H of $c_0(\Delta)$, $j(H) \leqq i(H)$. If H is a strong weakly compact set then, in fact, $j(H) = i(H)$. If the strong weakly compact set H is a continuous image of an* EC, *K, then $i(H) \leqq i(K)$.*

PROOF. Given $H \subset c_0(\Delta)$, fix a homeomorphic embedding $K \subset c_0(\Gamma)$ of H so that $i(H) = s(K)$. As H is a continuous image of K, Lemma 3.2 gives that $j(H) \leqq s(K) = i(H)$.

If H is a strong weakly compact set then by the first part $j(H) \leqq i(H)$ — and by Lemma 3.3 $i(H) \leqq j(H)$ so they are equal.

Finally, if H is a continuous image of K, then by choosing appropriate homeomorphic copy of K we can assume that $i(K) = s(K)$. By Lemma 3.2 $j(H) \leqq s(K) = i(K)$, and by the previous part of the Theorem $i(H) = j(H)$ when H is a strong weakly compact set.

4. Spaces with large index

Throughout this section we fix a cardinal τ satisfying $\tau^\omega = \tau$, and we construct a family $\{K_\xi : \xi < \tau^+\}$ of strong weakly compact sets of weight τ, whose indices $i(K_\xi)$ are unbounded in τ^+.

The sets K_ξ will be constructed by transfinite induction. For successor ordinals, and for ordinals with uncountable cofinality, the construction is very simple, and the spaces K_ξ will only satisfy $s(K_\xi) \geqq \xi$ for these ordinals. The crucial construction is when $\mathrm{cof}(\xi) = \omega$. It is in this case that the condition $\tau^\omega = \tau$ is used, and it is for these ordinals that we prove the desired stronger result that $i(K_\xi) \geqq \xi$.

We start with $K_1 = \{A \subset \Gamma : |A| \leqq 1\}$, where $|\Gamma| = \tau$. Then K_1 is homeomorphic to the one-point compactification of the discrete set Γ, and $P_1(K_1, 1)$ is the single point $\varnothing$, hence $s(K_1) = 1$. We now distinguish three cases.

Case I. $\xi = \eta + 1$

Assume K_η is an adequate strong weakly compact subset of $c_0(\Gamma)$, where $|\Gamma| = \tau$, and that $s(K_\eta) \geqq \eta$. Define

$$K_\xi = \{A \cup B : A \in K_\eta \text{ and } B \subset \Gamma,\ |B| \leqq 1\}.$$

The set K_ξ is an adequate strong weakly compact set, and for every ordinal $\alpha \leqq s(K_\eta)$

$$P_\alpha(K_\xi, 1) \supseteq \{A \cup B : A \in P_\alpha(K_\eta, 1) \text{ and } B \subset \Gamma,\ |B| \leqq 1\}$$

hence $s(K_\xi) \geqq s(K_\eta) + 1 \geqq \eta + 1 = \xi$.

Case II. $\mathrm{cof}(\xi) > \omega$

Assume that for each $\eta < \xi$, $K_\eta \subset c_0(\Gamma_\eta)$ are adequate strong weakly compact sets satisfying $s(K_\eta) \geqq \eta$. Let Γ be the disjoint union of the Γ_η's and define $K_\xi = \bigcup \{K_\eta : \eta < \xi\} \subset c_0(\Gamma)$. Topologically, K_ξ is obtained from the one point compactification of the disjoint union of the K_η's by identifying all the different points $\varnothing \subset \Gamma_\eta$ to the unique point $\varnothing \subset \Gamma$.

One checks easily that K_ξ is an adequate strong weakly compact set, and that $s(K_\xi) \geqq \sup\{s(K_\eta) : \eta < \xi\} \geqq \xi$.

Case III. $\mathrm{cof}(\xi) = \omega$

Assume $\xi_n \uparrow \xi$ and that for each n, K_n is an adequate strong weakly compact subset of $c_0(\Gamma_n)$ satisfying $s(K_n) \geqq \xi_n$ where $|\Gamma_n| = \tau$. Let $\Gamma = \Pi_{n=1}^\infty \Gamma_n$ and let $\pi_n : \Gamma \to \Gamma_n$ be the projection on the n-th coordinate. As $\tau^\omega = \tau$, $|\Gamma| = \tau$, and K_ξ is defined as a family of finite subsets of Γ as follows:

A finite subset $A \subset \Gamma$ belongs to K_ξ iff there is an $n \in \mathbf{N}$ so that

(1) $\pi_n(A) \in K_n$.

(2) *If $x \neq y$ are in A then $x(i) = y(i)$ for all $i < n$ and $x(n) \neq y(n)$.*

We say that A is witnessed by n.

It is clear that K_ξ is adequate. To see that it is weakly compact assume $A_1 \subset A_2 \cdots$ is an increasing chain in K_ξ with $|A_1| \geqq 2$. Fix n that witnesses A_1. If $x \neq y$ are in A_1, then n is the first coordinate where they differ, and since they belong to all the A_j's, the same n must witness all the A_j's. By (1) this means that $\pi_n(A_j) \in K_n$ for all j, and by (2) π_n is one-one on each A_j, i.e., $\pi_n(A_j)$ is an increasing chain in K_n — hence must be finite.

To estimate $j(K_\xi)=i(K_\xi)$, let (Δ_n) be an increasing chain with $\Gamma=\bigcup\Delta_n$. Considering each Γ_n with its discrete topology, and Γ with the product topology, Γ admits a complete metric, hence Baire's Category Theorem applies and there is a t so that the closure of Δ_t has non-empty interior. So find a basic open set $V=(\gamma_1,\dots,\gamma_k)\times\Pi_{k+1}^{\infty}\Gamma_n$ with $V\cap\Delta_t$ dense in V. We shall show that for each $m>k$, $K_\xi\,|_{\Delta_t}$ contains a subset which is combinatorically equivalent to K_m. Thus $s(K_\xi\,|_{\Delta_t})\geqq s(K_m)\geqq\xi_m$, and since m was arbitrary $s(K_\xi\,|_{\Delta_t})\geqq\xi$. As this holds for all chains (Δ_n) with $\Gamma=\bigcup\Delta_n$, this proves that $j(K_\xi)\geqq\xi$.

Thus fix $m>k$ and fix points $\gamma_j\in\Gamma_j$ for $k<j<m$ arbitrarily. For each $\gamma\in\Gamma_m$ consider the open set $V_\gamma=(\gamma_1,\dots,\gamma_{m-1},\gamma)\times\Pi_{m+1}^{\infty}\Gamma_n\subset V$. As Δ_t is dense in V, we can find for each $\gamma\in\Gamma_m$ a point $\sigma_\gamma\in V_\gamma\cap\Delta_t$. But now note that for each $A\in K_m$, $\tilde{A}=\{\sigma_\gamma:\gamma\in A\}$ belongs to K_ξ: It is witnessed by m. Thus $\tilde{K}_m=\{\tilde{A}:A\in K_m\}\subset K_\xi\,|_{\Delta_t}$ is combinatorically equivalent to K_m as required.

5. Sets with large index for $\tau=\omega_1$

In this section we construct a family $\{K_\xi:\xi<\omega_2\}$ of adequate strong weakly compact subsets of $c_0(\omega_1)$ satisfying $i(K_\xi)\geqq\xi$ for all $\xi<\omega_2$. It would be of interest to know if a similar construction is possible, without using any set theoretic assumptions, for all cardinals of uncountable cofinality.

The construction depends on the existence of uncountable almost disjoint families of infinite subsets of $\mathbf{N}$ (i.e., $|N_1\cap N_2|<\infty$ for different sets in such a family). In the construction of K_ξ when $\mathrm{cof}(\xi)=\omega$ any such family will do — but when $\mathrm{cof}(\xi)=\omega_1$ we shall need one with an additional property, and we first construct such a family.

LEMMA 5.1. *There is an almost disjoint family* $\{N_\alpha:\alpha<\omega_1\}$ *of infinite subsets of* $\mathbf{N}$ *so that for each* $\alpha<\beta<\omega_1$ *we have*

$$|\{\eta<\alpha:|N_\eta\cap N_\alpha|=|N_\eta\cap N_\beta|\}|<\infty. \tag{5.1}$$

PROOF. We formulated (5.1) as we are going to use it. In fact we construct by transfinite induction sets N_α satisfying the stronger condition that

$$|\{\eta<\beta:|N_\eta\cap N_\alpha|=|N_\eta\cap N_\beta|\}|<\infty. \tag{5.2}$$

Assume $\{N_\alpha:\alpha<\beta\}$ have already been chosen, and renumber them as $\{N_n:n<\omega\}$. We choose the set N_β by choosing successively $N_\beta\cap N_m$, $m=1,2,\dots$ and ensuring that

$$|N_n\cap N_m|\neq|N_m\cap N_\beta|\qquad\text{for all } n<m. \tag{5.3}$$

To do this assume $N_1 \cap N_\beta, \dots, N_{m-1} \cap N_\beta$ are already chosen. As N_m is almost disjoint from $N_1, \dots, N_{m-1}$ the set $N_m \setminus (\bigcup_{j=1}^{m-1} N_j)$ is infinite, and we can thus choose for $N_m \cap N_\beta$ a number of points from this set so as to ensure that $|N_\beta \cap N_m|$ is different from the $m-1$ numbers $\{|N_m \cap N_n| : n < m\}$. This proves (5.3), and (5.2) holds for these sets. Indeed given $\alpha < \beta < \omega_1$, consider the construction of N_β. At this step N_α was denoted by N_n for some n — and by the construction of N_β there are at most $n-1$ of the $\{N_\eta : \eta < \beta\}$, namely $N_1, \dots, N_{n-1}$ for which $|N_\eta \cap N_\beta| = |N_\eta \cap N_n|$ could be possible.

We now pass to the inductive construction of $\{K_\xi : \xi < \omega_2\}$. Starting with $K_1 = \{A \subset \omega_1 : |A| \leqq 1\}$ as in the previous section, the sets K_ξ will satisfy a property stronger than $j(K_\xi) \geqq \xi$, namely

(5.4) $\qquad s(K_\xi |_\Gamma) \geqq \xi \qquad$ for every uncountable subset $\Gamma \subset \omega_1$.

Case I. $\xi = \eta + 1$

We use the same construction as in the previous section. It is clear that if K_η satisfies (5.4) so does K_ξ.

Case II. $\mathrm{cof}(\xi) = \omega$

Fix $\xi_n \uparrow \xi$, and for each n an adequate strong weakly compact subset K_n of $c_0(\omega_1)$ satisfying (5.4) for ξ_n. Let $\{N_\alpha : \alpha < \omega_1\}$ be an almost disjoint family of infinite subsets of $\mathbf{N}$, and define K_ξ as follows:

A finite subset $A \subset \omega_1$ belongs to K_ξ iff $A \in K_n$ for some $n \leqq \min\{|N_\alpha \cap N_\beta| : \alpha, \beta \in A, \alpha \neq \beta\}$. We say that A is witnessed by n.

The set K_ξ is adequate, and it is weakly compact — for if (A_j) were an infinite increasing chain in K_ξ, choose $\alpha \neq \beta$ in A_1. Then $\alpha, \beta \in A_j$ for all j, and thus each of the A_j's is witnessed by some $n(j) \leqq |N_\alpha \cap N_\beta|$. By passing to an infinite subchain, we can assume they are all witnessed by the same n — contradicting the fact that K_n does not contain an infinite increasing chain.

It remains to check (5.4). Fix $\Gamma \subset \omega_1$, $|\Gamma| = \omega_1$. For each fixed n, one can find $\Gamma_n \subset \Gamma$, $|\Gamma_n| = \omega_1$ so that $|N_\alpha \cap N_\beta| \geqq n$ for all $\alpha, \beta \in \Gamma_n$. Indeed, there are only countably many n-subsets of $\mathbf{N}$ — so Γ_n can be chosen so that all the $\{N_\alpha : \alpha \in \Gamma_n\}$ contain the same n-subset of $\mathbf{N}$. But now *every* $A \in K_n |_{\Gamma_n}$ is witnessed by n, i.e., $K_n |_{\Gamma_n} \subset K |_{\Gamma_n}$, and since K_n satisfies (5.4) for ξ_n, we obtain $s(K_\xi |_\Gamma) \geqq s(K_\xi |_{\Gamma_n}) \geqq s(K_n |_{\Gamma_n}) \geqq \xi_n$. As n was arbitrary, $s(K_\xi |_\Gamma) \geqq \sup \xi_n = \xi$.

Case III. $\mathrm{cof}(\xi) = \omega_1$

Let $\{\xi_\alpha : \alpha < \omega_1\}$ increase to ξ, and for each $\alpha < \omega_1$, assume K_α is an adequate strong weakly compact subset of $c_0(\omega_1)$ satisfying (5.4) with respect to ξ_α. Let

$\{N_\alpha : \alpha < \omega_1\}$ be the almost disjoint family constructed in Lemma 5.1. Define K_ξ as follows:

A finite subset $A \subset \omega_1$ belongs to K_ξ iff there is an $\eta < \min(A)$ so that:
(1) $A \in K_\eta$.
(2) $|N_\eta \cap N_\alpha| = |N_\eta \cap N_\beta|$ *for all* $\alpha, \beta \in A$.
We say that A is witnessed by η.

The set K_ξ is adequate, and it is weakly compact — if (A_j) were an increasing infinite chain in K_ξ, choose any $\alpha < \beta$ in A_1, so $\alpha, \beta \in A_j$ for all j. By (5.1) $\{\eta < \alpha : |N_\alpha \cap N_\eta| = |N_\beta \cap N_\eta|\}$ is finite, and each A_j is witnessed by an element of this set. Hence by passing to an infinite subchain we can assume all the A_j's are witnessed by the same η — contradicting the weak compactness of K_η.

Let now $\Gamma \subset \omega_1$ be uncountable, and fix $\eta < \omega_1$. Then there is an n so that $\Gamma_1 = \{\alpha \in \Gamma, \alpha > \eta \text{ and } |N_\alpha \cap N_\eta| = n\}$ is uncountable — and every $A \in K_\eta |_{\Gamma_1}$ is witnessed by η, i.e., $K_\eta |_{\Gamma_1} \subset K_\xi |_{\Gamma_1}$. Thus $s(K_\xi |_\Gamma) \geqq s(K_\eta |_{\Gamma_1}) \geqq \xi_\eta$, and as η was arbitrary $s(K_\xi |_\Gamma) \geqq \xi$.

REMARK. All the Eberlein Compacts constructed in this and the previous section are non-uniform (see [BS]). In fact one can easily check that $i(K) \leqq \omega$ for every uniform Eberlein Compact K.

The construction in section 4 is in fact modeled on the construction of a non-uniform EC in [BS]. The one in this section is different, and yields a non-uniform EC of weight ω_1 without using CH. Another example was constructed by different methods by Leiderman and Sokolov in [LS], example 5.3.

6. Universal spaces

Throughout this section τ will be a fixed cardinal of countable cofinality, and τ_n a sequence of cardinals increasing to τ. We shall also assume that τ is a strong limit, i.e., $2^\alpha < \tau$ for all cardinals $\alpha < \tau$.

The proofs of both Theorems, C and D, are basically done by using the properties of τ to enumerate all spaces of weight (resp., density character) less than τ — and then putting all these spaces together in the proper way to obtain the universal space.

The following very simple Lemma uses the same saturation argument used in Lemma 3.1.

LEMMA 6.1. *Let $K \subset c_0(\Gamma)$, $|\Gamma| = \tau$, be weakly compact. Then there are* *subsets* $\Gamma_1 \subset \Gamma_2 \subset \cdots$ *of* Γ, $\bigcup \Gamma_n = \Gamma$ *with* $|\Gamma_n| = \tau_n$ *so that* $x|_{\Gamma_n} \in K$ *for all* $x \in K$ *and all* n.

PROOF. It is enough to show that given any $\Delta_1 \subset \Gamma$, there is a $\Delta \subset \Gamma$ containing Δ_1 so that $|\Delta| = |\Delta_1|$ and $K|_\Delta \subset K$. Define a sequence Δ_n inductively: Having defined Δ_n, let $K_n \subset K$ be such that $|K_n| \leqq |\Delta_n|$ and $K_n|_{\Delta_n}$ is dense in $K|_{\Delta_n}$. Then take $\Delta_{n+1} = \Delta_n \cup \{\text{supports of all } x \in K_n\}$. One easily checks that $\Delta = \bigcup \Delta_n$ works.

PROOF OF THEOREM C. We first define a sequence K_n of EC's of weight smaller than τ as follows. List all homeomorphic types of EC's of weight at most τ_1 as $\{K(\xi_1) : \xi_1 < \eta_1\}$, and let K_1 be the one-point compactification of the disjoint union of the K_{ξ_1}'s. Denote by p_1 the point of infinity in K_1. Note that $\eta_1 \leqq 2^{(2^{\tau_1})} < \tau$. Indeed, η_1 is bounded by the cardinality of all subsets of $c_0(\tau_1)$. Thus also $w(K_1) < \tau$.

To define K_2, fix first any $\xi_1 < \eta_1$, and consider all EC's of weight at most τ_2, say L, so that L contains a closed subset, homeomorphic to $K(\xi_1)$ as a retract of L. For each such L consider all homeomorphic embeddings $f: K(\xi_1) \to L$ so that $f(K(\xi_1))$ is a retract of L, and for each such f consider all retractions $r: L \to f(K(\xi_1))$. Enumerate all these triples (L, f, r) as $\{K(\xi_1, \xi_2) : \xi_2 < \eta_2\}$. (Thus the notation suppresses the dependence on the homeomorphism and on the retraction. But, in fact, each space L is repeated in the list as many times as there are homeomorphisms $f: K(\xi_1) \to L$ and retractions $r: L \to f(K(\xi_1))$. A specific homeomorphism and a specific retraction are associated with each $K(\xi_1, \xi_2)$.)

Let K_2 be the one point compactification of the disjoint union of all the $\{K(\xi_1, \xi_2) : \xi_1 < \eta_1, \xi_2 < \eta_2\}$ and let p_2 be the point of infinity in K_2. Again $w(K_2) < \tau$. Indeed, fixing $K(\xi_1)$ there are at most $2^{(2^{\tau_2})}$ possible L's. For each L there are at most $|L|^{|K(\xi_1)|} \leqq (2^{\tau_2})^{(2^{\tau_1})}$ homeomorphisms f, and for each f at most $|K(\xi_1)|^{|L|} \leqq (2^{\tau_1})^{(2^{\tau_2})}$ retractions.

We continue in the same way: Having defined K_{n-1} fix $K(\xi_1, \dots, \xi_{n-1})$ and list all EC's L, $w(L) \leqq \tau_n$ containing a copy of $K(\xi_1, \dots, \xi_{n-1})$ as a retract, all homeomorphic embeddings $f: K(\xi_1, \dots, \xi_{n-1}) \to L$ as a retract of L, and all retractions $r: L \to f(K(\xi_1, \dots, \xi_{n-1}))$ as $\{K(\xi_1, \dots, \xi_{n-1}, \xi_n) : \xi_n < \eta_n\}$. Take K_n to be the one point compactification of the disjoint union of $\{K(\xi_1, \dots, \xi_n) : \xi_1 < \eta_1, \dots, \xi_n < \eta_n\}$, with p_n as the point of infinity. As before $w(K_n) < \tau$.

DEFINITION. A point $(x_1, x_2, \dots) \in \Pi K_n$ is consistent if one of the following two possibilities holds.

(1) There are $\xi_n < \eta_n$ so that for all $n, x_n \in K(\xi_1, \ldots, \xi_n)$ and $f_n(x_n) = r_n(x_{n+1})$ where $f_n : K(\xi_1, \ldots, \xi_n) \to K(\xi_1, \ldots, \xi_{n+1})$ and $r_n : K(\xi_1, \ldots, \xi_{n+1}) \to f_n(K(\xi_1, \ldots, \xi_n))$ are the homeomorphism and retraction associated with $K(\xi_1, \ldots, \xi_{n+1})$.

(2) There is an N so that the conditions in (1) hold only for $n < N$, and so that $x_n = p_n$ for all $n \geqq N$.

We can now define the universal space $\mathcal{U}$:

$$\mathcal{U} = \left\{(x_1, x_2, \ldots) \in \prod K_n : (x_1, x_2, \ldots) \text{ is consistent}\right\}.$$

It is easy to check that $\mathcal{U}$ is a closed subset of ΠK_n. The latter, being the product of a countable number of EC's, is itself an EC, and its weight is $\sup w(K_n) = \tau$. Thus $\mathcal{U}$ is also an EC of weight τ.

Claim. *Given any Eberlein Compact K, $w(K) \leqq \tau$, there are $\xi_n < \eta_n$ so that K is homeomorphic to*

$$\mathcal{U}(\xi_1, \xi_2, \ldots) = \{(x_1, x_2, \ldots) \in \mathcal{U} : x_n \in K(\xi_1, \ldots, \xi_n) \quad \textit{for all } n\}.$$

Indeed, consider K as a subset of $c_0(\Gamma)$, $|\Gamma| = \tau$, and use Lemma 6.1 to find subsets $\Gamma_1 \subset \Gamma_2 \subset \cdots$ with $\Gamma = \bigcup \Gamma_n$, $|\Gamma_n| = \tau_n$ and $K|_{\Gamma_n} \subset K$. Now $w(K|_{\Gamma_1}) \leqq \tau_1$, so $K|_{\Gamma_1}$ appears in the list as some $K(\xi_1)$. Next, $w(K|_{\Gamma_2}) \leqq \tau_2$, so the triple consisting of $K|_{\Gamma_2}$, the identity embedding of $K|_{\Gamma_1}$ into $K|_{\Gamma_2}$ and the retraction of $K|_{\Gamma_2}$ onto $K|_{\Gamma_1}$ given by the restriction to Γ_1, appears in the list as some $K(\xi_1, \xi_2)$. The indices $\xi_3, \xi_4, \ldots$ are defined similarly. Identifying each $K|_{\Gamma_n}$ with the appropriate $K(\xi_1, \ldots, \xi_n)$, and using the fact that $\Gamma = \bigcup \Gamma_n$, we see that the map $x \to (x|_{\Gamma_1}, x|_{\Gamma_2}, \ldots)$ is a homeomorphism of K onto $\mathcal{U}(\xi_1, \xi_2, \ldots)$, proving the Claim.

It remains to see that each $\mathcal{U}(\xi_1, \xi_2, \ldots)$ is a retract of $\mathcal{U}$. So fix $\xi_1, \xi_2, \ldots$ and let $f_n : K(\xi_1, \ldots, \xi_n) \to K(\xi_1, \ldots, \xi_{n+1})$ be the homeomorphism associated with $K(\xi_1, \ldots, \xi_{n+1})$. For $\bar{x} = (x_1, x_2, \ldots) \in \mathcal{U}$, define $N(\bar{x})$ to be the largest n so that $x_n \in K(\xi_1, \ldots, \xi_n)$. So, in particular $N(\bar{x}) = 0$ is $x_1 \notin K(\xi_1)$ and $N(\bar{x}) = \infty$ if $\bar{x} \in \mathcal{U}(\xi_1, \xi_2, \ldots)$. Finally fix an arbitrary point $\bar{z} \in \mathcal{U}(\xi_1, \xi_2, \ldots)$ and define the retraction $r : \mathcal{U} \to \mathcal{U}(\xi_1, \xi_2, \ldots)$ as follows:

$$r(\bar{x}) = \begin{cases} \bar{z} & \text{if } N(\bar{x}) = 0, \\ (x_1, \ldots, x_N, f_N(x_N), f_{N+1}(f_N(x_N)), \ldots) & \text{if } N(\bar{x}) = N,\ 0 < N < \infty, \\ \bar{x} & \text{if } N(\bar{x}) = \infty. \end{cases}$$

Checking the continuity of r is routine once the following observation is made: If $\bar{x}(j) \to \bar{x}$, and $0 \leqq N(\bar{x}) < \infty$, then $N(\bar{x}(j)) = N(\bar{x})$ for large enough j. Indeed, this follows from the fact that each of the sets $K(\xi_1, \ldots, \xi_m)$ is clopen in K_m, for $m = 1, 2, \ldots, N(\bar{x}) + 1$. For the same reason, if $N(\bar{x}) = \infty$ then necessarily $N(\bar{x}(j)) \to \infty$.

PROOF OF THEOREM D. The proof is very similar to that of Theorem C, and we only sketch it. We define spaces X_n analogous to the K_n's in Theorem C: Given $X(\xi_1, \ldots, \xi_{n-1})$ we consider all WCG spaces Y, of density character at most τ_n, containing a norm one complemented, isometric copy of $X(\xi_1, \ldots, \xi_{n-1})$, all isometric embeddings $T : X(\xi_1, \ldots, \xi_{n-1}) \to Y$ as a norm one complemented subspace of Y, and all norm one projections $P : Y \to T(X(\xi_1, \ldots, \xi_{n-1}))$. We list them as $\{X(\xi_1, \ldots, \xi_{n-1}, \xi_n) : \xi_n < \eta_n\}$. Let $A = \bigcup\{(\xi_1, \ldots, \xi_n) : \xi_i < \eta_i,\ n = 1, 2, \ldots\}$.

A function f on A, satisfying $f(\xi_1, \ldots, \xi_n) \in X(\xi_1, \ldots, \xi_n)$ is called consistent if:

(1) *For each $(\xi_1, \ldots, \xi_{n+1}) \in A$, if $T_n : X(\xi_1, \ldots, \xi_n) \to X(\xi_1, \ldots, \xi_{n+1})$ and $P_n : X(\xi_1, \ldots, \xi_{n+1}) \to X(\xi_1, \ldots, \xi_n)$ are the embedding and projection associated with $X(\xi_1, \ldots, \xi_{n+1})$ then*

$$T_n f(\xi_1, \ldots, \xi_n) = P_n f(\xi_1, \ldots, \xi_{n+1}).$$

(2) *There is an N so that for all $n \geqq N$*

$$T_n f(\xi_1, \ldots, \xi_n) = f(\xi_1, \ldots, \xi_{n+1}).$$

The universal space B is the completion of the space of all consistent functions, under the norm $\| f \| = \sup\{ \| f(\xi_1, \ldots, \xi_n) \| : (\xi_1, \ldots, \xi_n) \in A\}$. It follows from the fact that τ is a strong limit, that B has density character τ.

Fix a sequence (β_n) with $\beta_n < \eta_n$ for all n. Given $(\xi_1, \ldots, \xi_n) \in A$, let $N = N(\xi_1, \ldots, \xi_n)$ be the first N so that $\xi_N \neq \beta_N$. A function f belongs to the subspace $B(\beta_1, \beta_2, \ldots)$ of B if

$$f(\xi_1, \ldots, \xi_{n+1}) = T_n f(\xi_1, \ldots, \xi_n) \qquad \text{for all } n \geqq N - 1 \text{ (when } N > 1),$$

or

$$f(\xi_1, \ldots, \xi_n) = 0 \qquad (\text{when } N = 1).$$

If X is a WCG space, with density character τ, it is isometric to some $B(\beta_1, \beta_2, \ldots)$. The proof is similar to the argument in the proof of Theorem C, when Lemma 6.1 is replaced by the following theorem of Amir and Lindenstrauss [AL]:

If X is a WCG *space with density character τ, there is a sequence of commuting norm one projections (P_n) on X, so that P_nX has density character τ_n, and so that $P_nx \to x$ for all $x \in X$.*

Finally a norm one projection $Q : B \to B(\beta_1, \beta_2, \dots)$ is defined by

$$(Qf)(\xi_1, \dots, \xi_n) = T_nT_{n-1} \cdots T_{N-1} f(\beta_1, \dots, \beta_{N-1}) \qquad (\text{if } N > 1)$$

and

$$Qf(\xi_1, \dots, \xi_n) = 0 \qquad (\text{if } N = 1).$$

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